Research Guru
Online Journal of Multidisciplinary Subjects (ISSN : 2349-266X)
Volume-16, Issue-1, June-2022
www.researchguru.net

Infinite Double Integral Representation for the Polynomial Set

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Abstract:

In the present paper, an attempt has been made to express a Infinite Double Integral Representation for the Polynomial set $D_n\{(x_K), y\}$. Many interesting new results may be obtained as particular cases on separating the parameters.

AMS subject classification: Special Function-33(C)

Keywords: Integral Representation, Hypergeometric functions, Lauricella function, Legendre polynomial.

I. Introduction :

We define the certain hypergeometric polynomial set of n-variables by means of generating functions.

$$\left(\xi + vt^e\right)^{-\frac{\sigma}{-}} F\begin{bmatrix}(a_r) & \\ & \mu_1 y^{-e_1} t^{e_1} \\ (b_s) & \end{bmatrix}$$

$$F\begin{bmatrix}(A_p);(\alpha_g);(\gamma_{u_k}) \\ \mu x_1^{t_1} t, \mu_2 x_2^{e_2} y^{-e_2} t^{e_2} \ldots\ldots\ldots \mu_k x_k^{e_k} t^{e_k} \\ (B_q);(\beta_h);(\delta_{v_k})\end{bmatrix}$$

$$\sum_{n=0}^{\infty} D_{n,e;e_1,e_2;.....e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v_k})}^{v;\xi,\sigma;\mu;\mu_1;\mu_2;.......\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{u_k});} \left\{(x_k),y\right\} t^n. \qquad (1.1)$$

Where $v,\xi,\sigma,\mu,\mu_1\mu_2,.......\mu_k$ are real and $e,e_1,e_2,.....e_k,r_1$ are non-negative integers.

The left hand sides of (1.1) contains the product of generalized hypergeometric functions involving Lauricella functions in the notation of Burchnall and Chaundy [1]. The generalized polynomial set contains a number of parameters, for simplicity we shall denote it

$$D_{n,e;e_1,e_2;.....e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v_k})}^{v;\xi,\sigma;\mu;\mu_1;\mu_2;.......\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{u_k});} \left\{(x_k),y\right\} by\ D_n\left\{(x_k),y\right\}.$$

Where *n* denotes the order of the generalized polynomial set $D_n\left\{(x_k),y\right\}$.

After little simplification (1.1) gives.

$$D_n\left\{(x_k),y\right\} = \xi^{\sigma} \sum_{m=0}^{\left[\frac{n}{e}\right]} \sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]} .... \sum_{m_k=0}^{\left[\frac{n-em-e_1m_1.......-e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times \frac{\left[(A_p)\right]_{n-em-e_1m_1-(e_2-1)m_2.......(e_k-1)m_k} \left[(\alpha_g)\right]_{n-em-e_1m_1-e_2m_2......e_km_k}}{\left[(B_q)\right]_{n-em-e_1m_1-(e_2-1)m_2.......(e_k-1)m_k} \left[(\beta_h)\right]_{n-em-e_1m_1-e_2m_2......e_km_k}}$$

$$\times \frac{\left[(\gamma_{u_k})\right]_{mk} \left[(a_r)\right]_{m_1} (\sigma_m)_m (-v)^m \mu_1^{m_1} \left(\mu_2 x_2^{e_2}\right)^{m_2} .... \left(\mu_k x_x^{e_k}\right)^{m_k}}{\left[(\delta_{v_k})\right]_{m_k} \left[(b_s)\right]_{m_1} \xi^m \quad m! \quad m_1! \quad y^{e_1m_1+e_2m_2} \quad m_2! \quad m_k!}$$

$$\times \frac{\left(\mu x_1^{r_1}\right)^{n-em-e_1m_1-e_2m_2-e_3m_3.....e_xm_k}}{(n-em-e_1m_1-e_2m_2-.......-e_km_k)!} \qquad (1.2)$$

2. Notations

(i)
$$(n) = 1,2....n-1,.$$
$$(a_p) = a_1,a_2,....a_p$$

(ii)
$$\left[\left(a_p\right)\right] = a_1, a_2, \ldots\ldots a_p.$$
$$\left[\left(a_p\right)\right]_n = \left(a_1\right)_n, \left(a_2\right)_n \ldots\ldots \left(a_p\right)_n.$$

(iii)
$$\Gamma\left[a + \frac{(m)}{m}\right] = \prod_{r=1}^{m} \Gamma\left(a + \frac{r}{m}\right).$$
$$\Gamma\left[a + \frac{(m) + \left(b_q\right)}{m}\right] = \prod_{r=1}^{m} \prod_{i=1}^{q} \Gamma\left(a + \frac{r + b_i}{m}\right).$$
$$\Gamma\left[\left(m : \left(a_p\right)\right)\right] = \prod_{i=1}^{p} \prod_{r=1}^{m} \Gamma\left(\frac{a_j + r - 1}{m}\right).$$
$$\Gamma\left[(a;b)\right] = \prod_{r=1}^{a} \Gamma\left(\frac{b + r - 1}{a}\right).$$

(iv)
$$\Gamma\left[a + \frac{(m)}{m}\right] = \prod_{r=1}^{m} \Gamma\left(a + \frac{r}{m}\right).$$
$$\Gamma\left[a + \frac{(m) + \left(b_q\right)}{m}\right] = \prod_{r=1}^{m} \prod_{i=1}^{q} \Gamma\left(a + \frac{r + b_i}{m}\right).$$
$$\Gamma\left[\left(m : \left(a_p\right)\right)\right] = \prod_{i=1}^{p} \prod_{r=1}^{m} \Gamma\left(\frac{a_j + r - 1}{m}\right).$$
$$\Gamma\left[(a;b)\right] = \prod_{r=1}^{a} \Gamma\left(\frac{b + r - 1}{a}\right).$$

(v)
$$\Gamma\left(a \pm b\right) = \Gamma\left(a + b\right)\Gamma\left(a - b.\right)$$
$$\Gamma_{**}\left(a + b\right) = \Gamma\left(a + b\right)\Gamma\left(a + b\right).$$

3. Theorem: For $e_2 > 1, \ldots\ldots, e_k > 1$

$$D_n\{(x_k), \mathcal{Y}\} = \frac{4\,\sigma_1^a \sigma_2^b \xi^\sigma}{\Gamma(\alpha)\Gamma\,(b)} \int_0^\infty \int_0^\infty e_1^{2a-1} \xi_1^{2b-1}\, e^{-\left(\sigma_1 e_1^2 + \sigma_2 \xi_1^2\right)}$$

$$\times F_{q+h+s\,:\,V_1,V_2,\ldots\ldots\ldots\ldots V_k}^{1+p+g+r\,:\,u_1,u_2\ldots\ldots\ldots u_k}\left[\begin{matrix}[-n; e, e_1, e_2 \ldots\ldots..e_k] \\ \underline{\hspace{4cm}},\end{matrix}\right.$$

$$[(1-(B_q)-n): e, e_1, e_2 - 1 \ldots e_k - 1], [(1-(\beta_h)-n): e, e_1, e_2 \ldots e_k],$$
$$[(1-(A_p)-n): e, e_1, e_2 - 1 \ldots e_k - 1], [(1-(\alpha_g)-n): e, e_1, e_2 \ldots. e_k],$$

$$[(\alpha_r) : 1], [(\gamma_{u_1}) : 1][(\gamma_{u_2}) : 1] \ldots [(\gamma_{u_k}) : 1], [\sigma: 1],$$
$$[(b_s) : 1], [(\delta_{V_1}) : 1], [(\delta_{V_2}) : 1] \ldots [(\delta_{V_k}) : 1],$$

$$\times \frac{-v(-1)^{e(r+s+p+q+g+h+1)}}{\xi(\mu x_1^{r_1})^{e}}, \frac{\mu_1(-1)^{e_1(r+s+p+q+g+h+1)}}{(\mu x_1^{r_1} y)^{e_1}},$$

$$\times \frac{\mu_2 x_2^{e_2}(-1)^{e_2(r+s+p+q+g+h+1)+p+q}}{(\mu x_1^{r_1} y)^{e_2}}, \ldots\ldots\ldots\ldots$$

$$\times \left.\frac{\mu_k x_k^{e_k}(-1)^{e_k(1+r+s+g+h+p+q+1)+p+q}}{(\mu x_1^{r_1})^{e_k}}\right] ded\xi \qquad \ldots\ldots(3.1)$$

Proof:

$$I \;=\; \xi^{\sigma} \int_0^{\infty}\int_0^{\infty} e_1^{2a-1}\xi_1^{2b-1}\, e^{-(\sigma_1 e_1^2+\sigma_2\xi_1^2)} \sum_{m=0}^{\left[\frac{n}{e}\right]} \sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_1}\right]} \ldots\ldots$$

$$\times \sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2\ldots\ldots e_{k-1}m_{k-1}}{e_k}\right]} \frac{[(A_p)]_{n-em-e_1m_1-(e_2-1)m_2\ldots\ldots(e_k-1)m_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2\ldots\ldots(e_k-1)m_k}}$$

$$\times \frac{[(\alpha_g)]_{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}[(a_r)]_m,[(\gamma_{u_1})]_{m_1}[(\gamma_{u_2})]_{m_2}\cdots[(\gamma_{u_k})]_{m_k}}{[(\beta_h)]_{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}[(b_s)]_m,[(\delta_{V_1})]_{m_1}[(\delta_{V_2})]_{m_2}\cdots[(\delta_{V_k})]_{m_k}}$$

$$\times \frac{(\sigma)_m(-v)^m\mu_1^{m_1}(\mu_2x_2^{e_2})^{m_2}\ldots\ldots(\mu_kx_k^{e_k})^{m_k}\sigma_1^{m_1}\sigma_2^{m_2}}{\xi^m\ \ m!\ m_1!\ y^{e_1m_1+e_2m_2}m_2!\ \ (m_k)!\ \ (a)_{m_1}(b)_{m_1}}$$

$$\times \frac{(\mu x_1^{r_1})^{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}}{(n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k)}de_1d\xi_1$$

$$= \xi^\sigma \sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\ldots\ldots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_{k-1}}{e_k}\right]}$$

$$\times \frac{[(A_p)]_{n-em-e_1m_1-(e_{2-1})m_2\ldots\ldots(e_k-1)m_k}[(\alpha_g)]_{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}}{[(B_q)]_{n-em-e_1m_1-(e_{2-1})m_2\ldots\ldots(e_k-1)m_k}[(\beta_h)]_{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}}$$

$$\times \frac{[(a_r)]_m,[(\gamma_{u_1})]_{m_1}[(\gamma_{u_2})]_{m_2}\cdots[(\gamma_{u_k})]_{m_k}(\sigma)_m(-v)^m\mu_1^{m_1}(\mu_2x_2^{e_2})^{m_2}}{[(b_s)]_m,[(\delta_{v_1})]_{m_1}[(\delta_{v_2})]_{m_2}\cdots[(\delta_{v_k})]_{m_k}\xi^m\ \ m!\,m_1!\ y^{e_1m_1+e_2m_2}}$$

$$\times \frac{\ldots\ldots(\mu_kx_k^{e_k})^{m_k}\sigma_1^{m_1}\sigma_2^{m_2}(\mu x_1^{r_1})^{n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k}}{m_2!\ m_k!\ (a)_{m_1}(b)_m(n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k)!}$$

$$\times \int_0^\infty\int_0^\infty e_1^{2a-1+2m_1\xi_1^{2b-1+2m_1}}e^{-(\sigma_1e_1^2\sigma_2\xi_1^2)}\,d_{e1}d_{\xi_1}$$

$$= \xi^\sigma \sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\ldots\ldots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2\ldots\ldots e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times \frac{[(A_p)]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}[(\alpha_g)]_{n-em-e_1m_1-e_2m_2........e_km_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}[(\beta_h)]_{n-em-e_1m_1-e_2m_2........e_km_k}}$$

$$\times \frac{[(\alpha_r)]_{m_1}[(\gamma_{u_1})]_{m_1}[(\gamma_{u_2})]_{m_2}\cdots[(\gamma_{u_k})]_{m_k}(\sigma)_m(-v)^m\mu_1^{m_1}(\mu_2x_2^{e_2})^{m_2}}{[(b_s)]_{m_1}[(\delta_{V_1})]_{m_1}[(\delta_{V_2})]_{m_2}\cdots[(\delta_{V_k})]_{m_k}\xi^m\ m!\ m_1!\ y^{e_1m_1+e_2m_2}}$$

$$\times \frac{......(\mu_kx_k^{e_k})^{m_k}(\mu x_1^{r_1})^{n-em-e_1m_1-e_2m_2........e_km_k}\sigma_1^{m_1}\sigma_1^{m_2}\Gamma(a+m_1)\Gamma(b+m_1)}{m_2!\ 4(a)_{m_1}(b)_m1\ m_k!\ (n-em-e_1m_1-e_2m_2........e_km_k)!\sigma_2^{a+m_2}}$$

$$= P\,\frac{\Gamma a\Gamma b}{4\ \sigma_1^a\sigma_2^b}\sum_{m,m_1,m_2,........m_k=0}^{\infty}\frac{[1-(B_q)-n]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}}{[1-(A_p)-n]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}}$$

$$\times \frac{[1-(\beta_h)-n]_{em+e_1m_1+e_2m_2........e_km_k}[(a_r)]_{m_1}[(\gamma_{u_1})]_{m_1}[(\gamma_{u_2})]_{m_2}\cdots[(\gamma_{u_k})]_{m_k}}{[1-(\alpha_g)-n]_{em+e_1m_1+e_2m_2........e_km_k}[(b_s)]_{m_1}[(\delta_{V_1})]_{m_1}[(\delta_{V_2})]_{m_2}\cdots[(\delta_{V_k})]_{m_k}}$$

$$\times \frac{(-n)_{em-e_1m_1-e_2m_2+.....+e_km_k}(-v)^m(-1)^{e(r+s+p+q+g+h+1)m}}{\xi^m\ m!\ (\mu x_1^{r_1})^{em}}$$

$$\times \frac{\mu_1^m(-1)^{e_1(r+s+p+q+g+h+1)m_1}}{(\mu x_1^{r_1}y)^{e_1m_1}},\frac{\mu_2x_2^{e_2}(-1)^{\{e_2(r+s+p+q+g+h+1)+p+q\}m_2}}{(\mu x_1^{r_1},y)^{e_2m_2}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,\frac{\mu_kx_k^{e_k}(-1)^{\{e_k(r+s+p+q+g+h+1)+p+q\}m_k}}{(\mu x_1^{r_1})^{e_km_k}} \qquad \ldots(3.2)$$

The Single terminating factor$(-n)_{em+e_1m_1+e_2m_2+\cdots..+e_km_k}$ makes all summation in (3.2) run upto ∞ and we finally arrive at

$$= P\frac{\Gamma(a)\Gamma(b)}{4\ \sigma_1^a\sigma_2^b}D_n\{(x_k),\mathcal{Y}\}$$

Hence the theorem

On using(156)

$$\int_0^\infty\int_0^\infty e^{2a-1}\xi^{2b-1}\,e^{-(\sigma_1e^2+\sigma_2\xi^2)}\boldsymbol{de}d\xi = \frac{\Gamma\,a\,\Gamma\,b}{4\ \sigma_1^a\sigma_2^b} \qquad \ldots(3.3)$$

Particular Cases of 3.2

(i) For p=o=q=g=h=u_2=v_2;σ=1=ξ=ν=r_1=y,μ=2=e_2;μ_2=-1 in (3.1), we get

$$H_n(x) = \frac{4\,\sigma_1^2\sigma_2^2(2x)^n}{n!\,\Gamma a\,\Gamma b}\int_0^\infty\int_0^\infty e_1^{2a-1}\,\xi_1^{2b-1}e^{-(\sigma_1e_1^2+\sigma_2\xi_1^2)}$$

$$\times F\left[\begin{matrix}\frac{-n}{2},\frac{-n}{2}+\frac{1}{2}; & \\ & -\frac{\sigma_1\sigma_2}{x^2}\\ (a:1),(b:1) & \end{matrix}\right]de_1d\xi_1$$

(ii) On taking p=0=q=g=h=u_2;v_2=1=ξ=σ=ν=r_1=e=δ,=y=μ;$\mu_2=\frac{1}{4}$, =2 and $\frac{x}{\sqrt{x^2-1}}$ for x_1 we achieve

$$P_n(x) = \frac{4\,\sigma_1^2\sigma_2^2x^n}{n!\,\Gamma a\,\Gamma b}\int_0^\infty\int_0^\infty e_1^{2a-1}\,\xi_1^{2b-1}e^{-(\sigma_1e_1^2+\sigma_2\xi_1^2)}$$

$$\times F\left[\begin{matrix} \frac{-n}{2},\frac{-n}{2}+\frac{1}{2}; \\ 1,(a:1),(b:1) \end{matrix} \frac{(x^2-1)}{x^2}\sigma_1\sigma_2\right] de_1 d\xi_1$$

(iii) On making the substitution p=o=q=u_2=v_2;h=1=y=ν=ξ=e=r_1=σ;e_2=2=μ;g={1,2}, μ_2=1;α_1=α; α_2=β;$=\beta_1=\alpha+\beta$ in (3.1) arrive at

$$G_n(\alpha,\beta;x)=\frac{4\ (\alpha)_n(\beta)_n(2x)^n\sigma_1^2 a_2^2}{n!\,(\alpha+\beta)_n\Gamma\, a\,\Gamma\, b}$$

$$\times F\left[\begin{matrix} \frac{-n}{2},\frac{-n}{2}+\frac{1}{2},1-\alpha-\beta-n; \\ 1-\alpha-n,1-\beta-n,(a:1),(b:1); \end{matrix} \frac{\sigma_1\sigma_2}{x^2}\right] de_1 d\xi_1$$

(vi) On putting q=0=h=u_2=p;ν_2=1=ξ=$\sigma=\nu$=r_1=e=y= ξ;g={1,2}, α_1=1=α_2= ν, δ_1= ν;e_2=2= μ;μ_2=-1 and z for x_1 in (3.1), we get

$$R_{n,\nu}\left(\frac{1}{2}\right)=\frac{4\ \sigma_1^a\sigma_2^b(\nu)_n(2Z)^n}{n!\,\Gamma\, a\,\Gamma\, b}\int_0^\infty\int_0^\infty e_1^{2a-1}\xi_1^{2b-1}e^{-\left(\sigma_1 e_1^2+\sigma_2\xi_1^2\right)}$$

$$\times F\left[\begin{matrix} \frac{-n}{2},\frac{-n}{2}+\frac{1}{2}; \\ -n,1-\nu-n.\nu,(a:1),(b:1); \end{matrix} \frac{\sigma_1\sigma_2}{z^2}\right] de_1 d\xi_1$$

(v) On putting q=o=h=y=u_2=v_2= σ;p=1=r_1=e= ν= ξ=j;e_2=3= μ;μ_2=-1$\beta_1=\nu$ and x_1=x in (3.1), we get

$$h_n(x)=\frac{4\ \sigma_1^a\sigma_2^b(\nu)_n(3x)^n}{n!\,\Gamma\, a\,\Gamma\, b}\int_0^\infty\int_0^\infty e_1^{2a-1}\xi_1^{2b-1}e^{-\left(\sigma_1 e_1^2+\sigma_2\xi_1^2\right)}$$

$$\times F\left[\begin{matrix} \Delta(3;-n); \\ \Delta(2;1-v-n),(a:1),(b:1); \end{matrix} \frac{\sigma_1\sigma_2}{4x^3}\right] de_1 d\xi_1$$

(vi) On putting , $a = \frac{1}{2}$, we achive

$$g_n^*(x) = \frac{4\left(\frac{1}{2}\right)_n \sigma_1^a \sigma_2^b (3x)^n}{n!\,\Gamma a\,\Gamma b} \int_0^\infty \int_0^\infty e_1^{2a-1} \xi_1^{2b-1} e^{-\left(\sigma_1 e_1^2 + \sigma_2 \xi_1^2\right)}$$

$$\times F\left[\begin{matrix} \Delta(3;-n) \\ \Delta\left(2;\frac{1}{2}-n\right),(a:1),(b:1); \end{matrix} \frac{\sigma_1\sigma_2}{4x^3}\right] de_1 d\xi_1$$

(vii)) On making the substitution p=o=q=u$_2$=v$_2$= σ ;=μ=1=r$_1$=y=v $=\xi$=e,e$_2$=3,μ$_2$=-1 and putting x$_1$=3y we get
μ_2=1;α_1=α; α_2=β;$=\beta_1 = \alpha+\beta$ in (3.1) arrive at

$$h_n^*(\mathcal{Y}) = \frac{4\,\sigma_1^a \sigma_2^b (3\mathcal{Y})^n}{} \int_0^\infty \int_0^\infty e_1^{2a-1} \xi_1^{2b-1} e^{-\left(\sigma_1 e_1^2 + \sigma_2 \xi_1^2\right)}$$

$$\times F\left[\begin{matrix} \frac{-n}{3},\frac{-n+1}{3},\frac{-n+2}{3}; \\ (a:1),(b:1); \end{matrix} \frac{\sigma_1\sigma_2}{\mathcal{Y}^3}\right] de_1 d\xi_1$$

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